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A note on coherence of dcpos [☆]

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Abstract

In this note, we prove that a well-filtered dcpo L is coherent in its Scott topology if and only if for every $x, y \in L$, $\uparrow x \cap \uparrow y$ is compact in the Scott topology. We use this result to prove that a well-filtered dcpo L is Lawson-compact if and only if it is patch-compact if and only if L is finitely generated and $\uparrow x \cap \uparrow y$ is compact in the Scott topology for every $x, y \in L$.

Keywords: coherence, well-filtered dcpo, Lawson compactness, patch topology

2000 MSC: 54B20, 06B35, 06F30

1. Introduction

In this paper, we investigate the *coherence* with respect to the Scott topology on directed-complete partial ordered sets (*dcpo*'s for short). Coherence, which states that the intersection of any two compact saturated sets is again compact, is an important property in domain theory [1, 3]. For instance, coherence is equivalent to Lawson compactness on pointed continuous domains [5]. This equivalence enabled the second author to characterise the Lawson compactness of continuous domains by the so-called “property M”, and use this element-level characterization to classify the category of continuous domains with respect to the cartesian closedness [5, 6].

In [9, 8], the equivalence between coherence and Lawson compactness was generalised to quasicontinuous domains. In Chapter 3 of [3], one even sees that on finitely generated quasicontinuous domains the compactness of $\uparrow x \cap \uparrow y$ for any $x, y \in L$, which seems much weaker than what coherence requires, already implies the Lawson compactness of L . In this note, we greatly generalize this result to *well-filtered* dcpos. Indeed, since every quasicontinuous domain is locally finitary compact and sober (see for example, [4]), our proof drops the locally finitary compact property and only uses well-filteredness, which is even strictly weaker than sobriety [7].

2. Preliminaries

We refer to [1, 3] for the standard definitions and notations of order theory and domain theory, and to [4] for topology.

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A topological space is called *well-filtered* if, whenever an open set U contains a filtered intersection $\bigcap_{i \in I} Q_i$ of compact saturated subsets, then U contains Q_i for some $i \in I$. Any sober space is well-filtered (see [3, Theorem II-1.21]). We take *coherence* of a topological space to mean that the intersection of any two compact saturated subsets is compact. A *stably compact* space is a topological space which is compact, locally compact, well-filtered and coherent. We call a dcpo L well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) if L with its Scott topology $\sigma(L)$ is a well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) space. Without further reference, we always equip L with the Scott topology $\sigma(L)$. Finally, a dcpo L is said to be *core-compact* if its Scott topology $\sigma(L)$ is a continuous lattice in the inclusion order.

For a topological space X , we denote the set of all compact saturated sets of X by $\mathcal{Q}(X)$. We consider the *upper Vietoris topology* v on $\mathcal{Q}(X)$, generated by the sets

$$\Box U = \{K \in \mathcal{Q}(X) \mid K \subseteq U\},$$

where U ranges over the open subsets of X . We use $\mathcal{Q}_v(X)$ to denote the resulting topological space. For a dcpo L , we use $\mathcal{Q}_v(L)$ to denote $\mathcal{Q}_v((L, \sigma(L)))$.

3. Main results

Lemma 3.1. *Let L be a well-filtered dcpo. Then L is coherent if and only if $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$.*

Proof. If L is coherent, it is obvious that $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$, since $\uparrow x, \uparrow y$ are compact saturated.

For the reverse, suppose $\uparrow x \cap \uparrow y$ is compact for all $x, y \in L$. We proceed to prove that for any compact saturated sets $A, B \subseteq L$, $A \cap B$ is compact in L . To this end, fix some element $a \in L$; we define a function f from L to $\mathcal{Q}_v(L)$ by sending an element x to the compact saturated set $\uparrow x \cap \uparrow a$. We claim that f is continuous. Indeed, for every Scott open subset $U \subseteq L$, $f^{-1}(\Box U) = \{x \mid \uparrow x \cap \uparrow a \subseteq U\}$ is obviously an upper set. Let $D \subseteq L$ be a directed subset with $\sup D \in f^{-1}(\Box U)$, then one has $\uparrow(\sup D) \cap \uparrow a \subseteq U$, that is, $\bigcap_{d \in D} (\uparrow d \cap \uparrow a) \subseteq U$. Note that L is well-filtered and $\{\uparrow d \cap \uparrow a \mid d \in D\}$ is a filtered family of compact saturated sets by assumption, so we have some $d \in D$ such that $\uparrow d \cap \uparrow a \subseteq U$, i. e., $d \in f^{-1}(\Box U)$. Hence f is continuous.

Since f is continuous, for the given compact saturated subset $A \subseteq L$, $f(A) = \{\uparrow x \cap \uparrow a \mid x \in A\}$ is a compact subset of $\mathcal{Q}_v(L)$. We now claim that the union of $f(A)$, which is just $A \cap \uparrow a$, is compact in L . Indeed, for any compact subset \mathcal{C} of $\mathcal{Q}_v(L)$, let $\{U_\alpha\}$ be a directed family of open sets of L covering $\bigcup \mathcal{C}$. By compactness, every element K of \mathcal{C} is already covered by one U_α ; in other words, $K \in \Box U_\alpha$. It follows that $\{\Box U_\alpha\}$ is a directed family covering \mathcal{C} , and now the compactness of \mathcal{C} tells us that $\mathcal{C} \subseteq \Box U_\alpha$ for some α . Hence $\bigcup \mathcal{C} \subseteq U_\alpha$ for this α . (This argument is similar to the one employed by Andrea Schalk in [10, Chapter 7] for showing that $\bigcup: \mathcal{Q}_v(\mathcal{Q}_v(X)) \rightarrow \mathcal{Q}_v(X)$ is well-defined.)

Now for such A the above argument enables us to define another function g from L to $\mathcal{Q}_v(L)$ as: $g(x) = \uparrow x \cap A$ for every $x \in L$. A similar deduction shows that g is continuous. So for the compact saturated subset B of L , $g(B)$ is compact in $\mathcal{Q}_v(L)$, and again the union of $g(B)$, which is $A \cap B$, is compact in L . So L is coherent. \square

Corollary 3.2. *Every well-filtered complete lattice L is coherent.*

Proof. For every $x, y \in L$ the intersection of $\uparrow x$ and $\uparrow y$, which is $\uparrow(x \vee y)$, is always compact, so the statement follows from Lemma 3.1. \square

The following fact about core-compact complete lattices is essentially due to G. Gierz and K.H. Hofmann [2]; we collect it here as a corollary to the previous result.

Corollary 3.3. *For a complete lattice L , the following statements are equivalent:*

1. L is core-compact, i. e., $\sigma(L)$ is a continuous lattice;
2. $(L, \sigma(L))$ is stably compact.

Proof. The only interesting part is that 1 implies 2. Suppose L is a complete lattice and $\sigma(L)$ is continuous, then $(L, \sigma(L))$ is a locally compact sober space by [3, Proposition VII-4.1]. Since sober spaces are well-filtered, L is coherent by Corollary 3.2. Finally, L is obviously compact in its Scott topology since it has a least element. \square

We now come to a characterization of the compactness of Lawson and patch topologies on L . Recall that the patch topology on L arises by taking all Scott closed sets together with all compact saturated sets as a subbasis for the closed sets; whereas the (coarser) Lawson topology is generated by the Scott closed subsets and principal upper sets $\uparrow x$. The following theorem is a generalization of [3, Theorem III-5.8] which is stated for quasicontinuous domains.

Theorem 3.4. *Let L be a well-filtered dcpo. Then the following statements are equivalent:*

1. L is patch-compact, i. e., L is compact in the patch topology;
2. L is Lawson-compact;
3. L is compact and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
4. L is finitely generated and $\uparrow x \cap \uparrow y$ is compact for every $x, y \in L$;
5. L is finitely generated and coherent.

Proof. (1 \Rightarrow 2): That 1 implies 2 is true for all dcpos since the patch topology is finer than the Lawson topology.

(2 \Rightarrow 3): It is obvious that L is compact since the Lawson topology is finer than the Scott topology. For every $x, y \in L$, $\uparrow x \cap \uparrow y$ is Lawson closed; therefore it is Lawson-compact, thus Scott compact.

(3 \Rightarrow 4): Since L is compact, by the Hausdorff Maximality Principle, every element is above some minimal element of L . Denote the set of all minimal elements of L by M ; M must be finite. Otherwise, the family $\{M \setminus F \mid F \subseteq_{\text{fin}} M\}$ is a filtered set of non-empty Scott closed sets with an empty intersection, which contradicts compactness.

(4 \Rightarrow 5): This is from Lemma 3.1.

(5 \Rightarrow 1): This is a straightforward consequence of [3, Lemma VI-6.5]¹. \square

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¹This lemma works for sober dcpos in [3]. However, one can find that its proof only uses well-filteredness.

References

- [1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Semantic Structures*, volume 3 of *Handbook of Logic in Computer Science*, pages 1–168. Clarendon Press, 1994.
- [2] G. Gierz and K. H. Hofmann. On complete lattices L for which $O(L)$ is continuous – A lattice theoretical characterisation of CS . Available at <http://www.mathematik.tu-darmstadt.de/~keimel/SCS/scs34.pdf>, August 1977.
- [3] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003.
- [4] J. Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory*, volume 22 of *New Mathematical Monographs*. Cambridge University Press, 2013.
- [5] A. Jung. *Cartesian Closed Categories of Domains*, volume 66 of *CWI Tracts*. Centrum voor Wiskunde en Informatica, Amsterdam, 1989. 107 pp.
- [6] A. Jung. The classification of continuous domains. In *Proceedings, Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 35–40. IEEE Computer Society Press, 1990.
- [7] Hui Kou. Uk-admitting dcpo’s need not be sober. In K. Keimel, G.-Q. Zhang, Y.-M. Liu, and Y.-X. Chen, editors, *Domain and Processes*, volume 1 of *Semantic Structures in Computation*, pages 41–50. Kluwer Academic Publishers, 2001.
- [8] Hui Kou, Ying-Ming Liu, and Mao-Kang Luo. On meet-continuous dcpos. In G. Zhang, J. Lawson, Y. Liu, and M. Luo, editors, *Domain Theory, Logic and Computation*, volume 3 of *Semantic Structures in Computation*, pages 117–135. Springer Netherlands, 2003.
- [9] J. Lawson. The upper interval topology, Property M, and compactness. In Abbas Edalat, Achim Jung, Klaus Keimel, and Marta Kwiatkowska, editors, *Comprox III, Third Workshop on Computation and Approximation*, volume 13 of *Electronic Notes in Theoretical Computer Science*, pages 158–172. Elsevier Science Publishers, 1998.
- [10] A. Schalk. *Algebras for Generalized Power Constructions*. Doctoral thesis, Technische Hochschule Darmstadt, 1993. 166 pp.